Open-loop and closed-loop filters for optimal trajectory planning under time and frequency constraints

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Abstract— Dynamic filters for real-time trajectory generation can be designed in different ways with quite different levels of performances and complexity. However, one can observe that the configurations of such filters are based on two main schemes: systems composed of a chain of integrators with a feedback control and systems formed of a sequence of linear filters disposed in a cascade configuration. According to these schemes, it is possible to obtain minimum-time trajectories under constraints of velocity, acceleration, jerk and even higher derivatives. In both cases, the degree of continuity of the resulting trajectory depends on the order $n$ of the filter, which can be designed according to a modular approach.

After a short overview of the structure of the trajectory generators, pro and cons of the two approaches are analyzed. In particular, the attention is focussed on linear filters, since their structure allows a straightforward characterization of the trajectory from a frequency point of view. As a consequence, the generator can be designed by taking into account frequency constraints, besides more standard time constraints (i.e. limits on velocity, acceleration, etc.). The proposed method combines the advantages of minimum-time trajectories with those of input shaping techniques. Moreover, it is possible to prove that, under additional hypotheses, the same chain of linear filters proposed for minimum-time trajectories generation can be used for obtaining uniform B-spline curves, that are widespread in the robotic field when the interpolation of a set of given via-points is required. In this case, the additional constraints do not allow to impose limits on the velocity or acceleration, but only to properly shape the trajectory in the frequency domain. It is therefore possible to select the trajectory/filter parameters with the purpose of suppressing residual vibrations, that may be present because elastic phenomena affecting the robotic system.

I. INTRODUCTION

Online generation of trajectories subject to kinematic constraints (on velocity, acceleration, jerk, etc.) plays a central role in all those applications where the motions cannot be planned a priori and must be optimized with respect to the time. Robotic systems are probably the most important example of such applications because their flexibility and the complexity of the required movements. For this reason, a large number of papers addressing this problem is available in the scientific literature tied to the robotic field, both for single-axis applications and for multi-axis motions. With respect to this problem, several filters based on control theory have been proposed, see e.g. [1], [2], [3], [4] among many others. These trajectory generators are based on two opposite design philosophies, i.e. closed-loop and open-loop approaches. Both allow to obtain minimum-time trajectories compliant with given limits on velocity, acceleration, jerk, etc., by specifying in runtime the desired final position. Additionally, the structure of the dynamic filters has relevant implications on the spectral content of the motion profile and can be properly modified in order to take into account frequency specifications and not only time-domain constraints.

As a matter of fact, the need of high velocities often leads to the excitation of eigenfrequencies of the machines/robots caused by structural flexibilities and may produce vibrations and large tracking errors. For this reason, a number of works copes the problem of filtering preplanned trajectories in order to reduce residual vibrations. The available methods range from low-pass and notch filters to input shaping techniques, see [5] for a comparative overview, but only recently an online generator, based on a chain of linear filters, that combines the advantages of minimum-time trajectories with those of shaping techniques has been proposed [6].

With some additional constraints on the free parameters of the filters, it is possible to show that open-loop generators for minimum-time trajectories share the same structure of generators for B-splines curves which are extensively used in robotics in order to define smooth trajectories crossing a set of given via-points. Therefore, the considerations and the techniques used for properly shaping the spectral content of minimum-time motions can be extended to this class of curves.

The paper is organized as follows. In Sec. II the main concepts tied to time-optimal trajectories and dynamic filters for online trajectory generation are presented, and a general overview on closed-loop and open-loop filters is provided. Then, in Sec. III the two different types of trajectory generators are compared with respect to the different problems that they can solve (point-to-point optimal trajectory generation, smoothing of pre-planned trajectories, etc.). In Sec. IV open-loop filters are analyzed in the frequency domain and their parameters are set with the purpose of properly shaping the spectrum of the output trajectory. Similar considerations are reported in Sec. V with respect to B-spline trajectory generation. Concluding remarks are reported in the last section.

II. OPTIMAL TRAJECTORIES AND DYNAMIC FILTERS

The optimization process of trajectories subject to constraints on velocity, acceleration, jerk, etc., leads to the so-called multi-segment trajectories, i.e. trajectories composed by several tracts properly joined, each one characterized by a specific analytical expression, and in which the velocity, the
one obtains a trajectory \( q_n(t) \) of class \( C^{n-1} \), that is with the
first \( n - 1 \) derivatives that are continuous, while the \( n \)-th derivative \( q_n^{(n)}(t) \) is a piece-wise constant function whose
values belong to a set \( \{q_{\text{min}}, 0, q_{\text{max}}\} \). The number \( n \) is
called order of the trajectory. The dynamic filters for trajec-
tory planning generate on-line a time optimal trajectory \( q_n(t) \)
that tracks at best a reference signal \( r(t) \), satisfying desired
constraints on the first \( n \) derivatives of \( q_n(t) \). The reference
signal \( r(t) \) is generally given by a first coarse trajectory
generator providing for instance a piecewise constant profile
which defines the desired final positions, or is an external
input, given for example by a human operator.

The trajectory planners based on feedback regulation are
composed of a chain of \( n \) integrators and a nonlinear
controller able to nullify in minimum time the tracking error
between the reference input \( r(t) \) and the integrators output
\( x_n(t) = q_n(t) \), being compliant with constraints (1), see
Fig. 1. In [7] a modular solution is proposed. According to
this approach, the \( n \)-th order trajectory planner is designed
around the filter of order \( n - 1 \), as illustrated in Fig. 2. The

structure of the \( i \)-th controller is

\[
C_i : u_i = \begin{cases} 
\pi_{i-1}, & \text{if } y_{i-1} \leq h_i(y_{i-1}, y_{i-2}, \ldots, y_1) \\
0, & \text{if } y_{i-1} = y_{i-2} = \ldots = y_1 = 0 \\
\xi_{i-1}, & \text{if } y_{i-1} \geq h_i(y_{i-1}, y_{i-2}, \ldots, y_1)
\end{cases}
\]

where \( x_i \) is the output of the filter, that is the position
trajectory, \( r_i \) is the reference signal that denotes the target
position and \( y_i = x_i - r_i \) is the tracking error. The parameters
\( \pi_{i-1}, \xi_{i-1} \) denote the limits on the first derivative of \( x_i \).

While the structure of \( C_i \), which is based on variable
structure control, is rather simple, the expression of the function
\( h_i(\cdot) \) that appears in (2) is very complicated also
for small values of the index \( i \), see [7] for the detailed
expression. Moreover, the computation of \( h_i(\cdot) \) for \( i > 3 \)
is critical and has not been performed yet. Therefore, at
the moment, only second- or third-order trajectories may
be generated according to this approach. Finally, the digital
implementation of the trajectory generator starting from its
continuous-time expression is not straightforward and cannot
be obtained by a simple discretization of the integrators chain
since the filter will be certainly affected by chattering. For
this reason, ad hoc solutions are required.

Open-loop trajectory generators are characterized by a very
simple structure which allows a generalization up to what-
ever order \( n \) with only a little increase of complexity and
computational burden. As a matter of fact, a multi-segment
trajectory of order \( n \) can be obtained by filtering a step input
with a cascade of \( n \) dynamic filters, each one characterized
by the transfer function

\[
M_i(s) = \frac{1 - e^{-sT_i}}{sT_i}
\]

where the parameter \( T_i \) (in general different for each filter
composing the chain) is a time length, see Fig. 3. In
mathematical terms, this means that

\[
q_n(t) = h \cdot u(t) \ast m_1(t) \ast m_2(t) \ast \ldots \ast m_n(t)
\]

where \( u(t) \) denotes the unit step function, \( h \) is the desired
displacement and \( m_i(t) = C^{i-1}\{M_i(s)\} \) is the impulse
response of each filter. The parameters \( T_i \) can be selected
with the purpose of imposing desired bounds on velocity,
acceleration, jerk and higher derivatives, i.e.

\[
|q_n^{(i)}(t)| \leq q_{\text{max}}^{(i)}, \quad i = 1, \ldots, n
\]

by assuming

\[
T_1 = \frac{|h|}{q_{\text{max}}^{(1)}}, \quad T_i = \frac{|h|}{q_{\text{max}}^{(i-1)}}, \quad i = 2, \ldots, n
\]

\[
h \cdot u(t) \frac{1 - e^{-sT_1}}{sT_1} q_1(t) \frac{1 - e^{-sT_2}}{sT_2} q_2(t) \ldots \frac{1 - e^{-sT_n}}{sT_n} q_n(t)
\]

Fig. 1. Structure of a closed-loop trajectory filter of \( n \)-th order.

Fig. 2. Structure of the \( i \)-th control loop \( (S_0 = 1) \) of the modular closed-
loop trajectory generator.

Fig. 3. System composed by \( n \) filters for the computation of an optimal
trajectory of class \( C^{n-1} \).
with the additional constraints
\[ T_j \geq T_{j+1} + \ldots + T_n, \quad j = 1, \ldots, n - 1. \] (7)

For more details refer to [6].

In order to evaluate the trajectory at discrete time instants \( kT_s \), being \( T_s \) the sampling period, the system composed by \( n \) filters may be discretized by applying on each filter \( M_i(s) \) the backward differences method that leads to the expression of a moving average filter
\[ M_i(z) = \frac{1}{N_i} \frac{1 - z^{-N_i}}{1 - z^{-1}} \] (8)

where \( N_i = T_i/T_s \) is the number of samples (not null) of the filter response. Note that \( N_i \) is also equal to the number of elements composing the FIR filter (usually called taps) as they appear in the equivalent (nonrecursive) formulation
\[ M_i(z) = \frac{1}{N_i} + \frac{1}{N_i} z^{-1} + \frac{1}{N_i} z^{-2} + \ldots + \frac{1}{N_i} z^{-N_i-1}. \] (9)

The implementation of the proposed trajectory generator on a digital controller can be achieved by simply considering the function \( M_i(z) \) in lieu of the corresponding function \( M_i(s) \) in the block-scheme of Fig. 3. Note that the digital implementation of each filter only requires two additions and one multiplication. As a consequence, even for high values of the degree \( n \), the trajectory generator (composed by \( n \) filters) results very efficient from a computation point of view.

III. A COMPARISON BETWEEN CLOSED-LOOP AND OPEN-LOOP FILTERS

Besides the different structure, the two groups enjoy peculiar features that make each type of generator preferable for a specific application. Closed-loop filters are superior to analogous open-loop systems in terms of performances and flexibility, since they allow to take into account asymmetric bounds, which can even modified in runtime. Moreover, they do not require additional constraints among the filter parameters such as (7). On the other hand, they are affected by some limitations: presence of chattering superimposed to the output, high complexity of the implementation, high computational burden, maximum order \( n = 3 \). For these reasons the choice of a particular type of filter must be performed according to the specific application to be carried out. For instance, in standard tasks where online generation of point-to-point trajectories is required, if the desired bounds of velocity, acceleration, jerk, etc. meet the conditions (7), open-loop filters are preferable. As a matter of fact, as reported in Fig. 4(a) and Fig. 4(b) the two generators provide the same trajectory but with very different costs in terms of computational complexity. Moreover, in Fig. 4(a) one can observe the chattering in the jerk profile and the overshoots in the acceleration. Figure 4(c) shows the fourth order trajectory \( q_4(t) \) obtained by applying the same stepwise input function to a chain of four running average filters. With a little increase of complexity, one can obtain the time-optimal trajectory with the desired degree of smoothness.

On the other hand, if the input function changes before that the previous tract of trajectory has ended, the open-loop generator cannot guarantee the compliance with the constraints, see Fig. 5(b). On the contrary, closed-loop filters always fulfill the prescribed constraints, as shown in Fig. 5(a). Moreover, as already mentioned, in case of feedback controlled trajectory generators, it is possible to consider asymmetric limits, i.e. \( q_{max} \neq -q_{min} \), or even change the values of the bounds in runtime. In this case, the controller will act so that the new limits are satisfied in minimum time. For instance, in Fig. 6(b) the behavior of the filter is shown when the minimum value of velocity is increased from \(-0.2\) to \(-0.1\). As soon as this occurs, the controller modifies the velocity of the motion profile in order to meet the new limit but without violating the other constraints on acceleration and jerk. Conversely, by adopting an open-loop generator it is not possible to change the constraints during the planning of a trajectory, since this would require a structural modification of the filters composing it, leading to discontinuities of the response. In fact, the number of taps of the FIR filters is directly related to the desired bounds by means of (6).

Finally, both closed- and open-loop filters can be used to modify pre-planned trajectories with the purpose of making them compliant with the desired bounds and not only to generate point-to-point multi-segment trajectories. Also in this case the two types of dynamic systems behave quite
differently. In Fig. 7 the responses to a reference input \( r(t) \) composed by ramp functions (therefore with constant velocity and impulsive acceleration) are reported. The closed-loop filters tries to reach in minimum-time and then to track the reference \( r(t) \), satisfying the given constraints. On the contrary, the third order open-loop filter produces a smoother but delayed version of the input. Note that the output signal reaches the constant velocity of the ramps but that all the other constraints are not reached. Moreover, since the input signal is continuous the resulting trajectory has continuous jerk. Therefore it is possible to consider a lower order trajectory filter (second-order), with the parameters

\[
T_1 = \frac{q_{max}^{(1)}}{q_{max}}, \quad T_2 = \frac{q_{max}^{(2)}}{q_{max}}
\]

and of the numerator \( Q_n(s) \), being the restriction to the imaginary axis, i.e. \( Q_n(j\omega) \). Therefore, the closed-form expression of \( Q_n(j\omega) \) is given by the products of the frequency responses of the filters composing the trajectory generator:

\[
Q_n(j\omega) = \frac{h}{j\omega} \cdot M_1(j\omega) \cdot M_2(j\omega) \cdot \ldots \cdot M_n(j\omega)
\]

As a matter of fact, as it is well known, the Fourier transform of \( q_n(t) \) immediately descends from \( Q_n(s) \), being the restriction to the imaginary axis, i.e. \( Q_n(j\omega) \). Therefore, the closed-form expression of \( Q_n(j\omega) \) is given by the products of the Fourier signal corresponding to the input \( h u(t) \) and of the frequency responses of the filters composing the trajectory generator:

\[
M_i(j\omega) = \frac{1 - e^{j\omega T_i}}{j\omega T_i} \frac{\sin \left( \frac{\omega T_i}{2} \right)}{\frac{\omega}{2}}.
\]

Since the frequency characterization of the trajectory, including its derivatives is a useful tool to predict vibratory phenomena in the systems to which the trajectory is applied.
[9], it is necessary to obtain the expression of the spectrum of the generic \( k \)-th derivative of \( q_n(t) \). Because of the properties of Laplace transforms, this result is straightforward. As a matter of fact, the Laplace transform of \( q_n^{(k)}(t) \) is given by
\[
Q_n^{(k)}(s) = s^k Q_n(s)
\]
and therefore the expression of the spectrum of \( q_n^{(k)}(t) \) is
\[
Q_n^{(k)}(j\omega) = (j\omega)^k Q_n(j\omega) = h \cdot (j\omega)^{k-1} M_1(j\omega) \cdot M_2(j\omega) \cdots M_n(j\omega).
\]

In conclusion, the amplitude spectrum of \( q_n(t) \) and its derivatives, i.e. \( |Q_n^{(k)}(j\omega)| \), is given by the product of two main elements:

- a power of \( \omega \), i.e. \( \omega^{k-1} \), being \( k \) the order of the derivative;
- the (magnitude of the) frequency response of the chain of \( n \) filters \( M_i(s) \).

The frequency response of the cascade of filters is the product of the single frequency responses \( M_i(j\omega) \), \( i = 1, \ldots, n \), whose magnitude is
\[
|M_i(j\omega)| = \left| \frac{\sin \left( \frac{\omega T_i}{\omega_i} \right)}{\frac{\omega T_i}{\omega_i}} \right| = \left| \frac{\omega}{\omega_i} \right|
\]
where \( \sin(\cdot) \) denotes the normalized sinc function defined as \( \sin(\pi x) = \frac{\sin(\pi x)}{\pi x} \) and \( \omega_i = \frac{2\pi}{T_i} \). Note that the function \( |M_i(j\omega)| \), shown in Fig. 13, is equal to zero for \( \omega = k\omega_i \), with \( k \) integer. This property can be profitably exploited to properly choose the parameters of the trajectory/filter with the purpose of nullifying the spectrum of the trajectory at critical frequencies, for instance the eigenfrequencies of the plant. For this aim, if \( \omega_i \) denotes a resonant frequency, it is sufficient to assume
\[
\omega_i = \frac{\omega_r}{l} \quad \Leftrightarrow \quad T_i = l \frac{2\pi}{\omega_r}, \quad l = 1, 2, \ldots \quad (12)
\]
This result generalizes what has been presented in [10] where, with reference to a double S velocity trajectory, it is recognized that in order to suppress residual vibrations due to the dominating vibratory mode of an axis of motion it is necessary to assume that the duration of the “jerk period” (in which the jerk remains constant) equals a multiple of the natural period of the vibrational mode. According to (12) the reduction of residual vibrations caused by resonant frequencies of the plant can be achieved with multi-segment trajectories of any order provided that the time constant \( T_i \) of a filter \( M_i(s) \) is \( l \) times, \( l \) integer, the dominating natural period \( T_0 = \frac{2\pi}{\omega_r} \).

For instance, if a standard motion system composed of two inertias with an elastic transmission lightly damped [11], [12], [13] is considered, some parameters of the trajectory generator can be selected with the purpose of minimizing frequency components about the resonance of the system. In Fig. 10 and Fig. 11, the responses of the system to two different third-order trajectories are reported, and in particular residual vibrations \( \varepsilon(t) \) are analyzed. When the parameter \( T_3 \) of the third filter of the chain is assumed equal to \( T_0 \) vibrations at the end of motion are canceled, see Fig. 10. On the contrary, if the resonant frequency \( \omega_r \) is not considered, the motion system may be affected by residual vibrations as shown in Fig. 11. Note that in this example, frequency constraints are taken into account together with time-constraints on velocity and acceleration. For more details refer to [6].

V. B-SPLINE FILTERS

An interesting property of the filter of Fig. 3 is the possibility of generating online B-spline trajectories. As a matter of fact, in [14], it has been shown that uniform B-splines\(^1\) of degree \( p \) can be computed by feeding a cascade of \( p \) filters
\[
M(s) = \frac{1}{T} \frac{1 - e^{-sT}}{s}, \quad (13)
\]
\(^1\)Uniform B-spline are defined as
\[
s(t) = \sum_{j=0}^{n} p_j B^p(t - jT), \quad 0 \leq t \leq (m-1)T,
\]
shown in Fig. 12, with the function

\[ p(t) = \sum_{j=0}^{n} p_j B^0(t - jT) \]

where \( p_j \) are the control point defining the B-spline curve (for their computation refer to [14] and [15]) and \( B^0(t) \) is a rectangular function defined as

\[ B^0(t) = \begin{cases} 
1, & \text{if } 0 \leq t < T \\
0, & \text{otherwise}.
\end{cases} \]

By comparing the trajectory generator of Fig. 3 with that of Fig. 12, one can immediately infer that the two systems are equivalent provided that \( T_1 = T_2 = \ldots = T_n = T \). Therefore, the two filters enjoy the same properties and, in particular, both allow to precisely characterize the motion profiles from a frequency point of view.

In terms of Laplace transform, uniform B-splines based on linear filters can be written as

\[ S^p_u(s) = \mathcal{L} \left\{ \sum_{j=0}^{n} p_j B^0(t - jT) \right\} \cdot M(s) \cdot M(s) \cdot \ldots \cdot M(s) = P(s) \cdot M^p(s) \]  

Therefore, the closed form expression of B-spline spectrum, that is \( S^p_u(j\omega) \), is given by the products of the Fourier signal corresponding to the input \( p(t) \) and the frequency response of the filters composing the trajectory generator:

\[ S^p_u(j\omega) = P(j\omega) \cdot M^p(j\omega). \]

Note that in this case, all the filters composing the chain have the same frequency response since they depend on a unique free parameter \( T \). The magnitude of the frequency response of the cascade of \( p \) filters, shown in Fig. 13 for \( p = 1, 2, 3, 4 \), is given by

\[ |M^p(j\omega)| = \left| \operatorname{sinc} \left( \frac{\omega}{\omega_0} \right) \right|^p \]

where \( \omega_0 = \frac{2\pi}{\omega} \). Like in case of multi-segment trajectories, the function \( |M^p(j\omega)| \) is equal to zero for \( \omega = k \omega_0 \). Moreover, by augmenting the value of the degree \( p \), the spectral components that follows the frequency \( \omega_0 \) are considerably reduced. These features of the spectrum of \( M^p(s) \) can be profitably exploited to properly choose the free parameter

where the vectorial coefficients \( p_j, j = 0, \ldots, m \), called control points, determine the shape of the curve. \( B^p(t) \) are B-spline basis functions of degree \( p \), and \( T \) denotes the (constant) time-distance between successive knots, i.e. \( t_{j+1} - t_j = T, \ j = 0, \ldots, m - 2 \).

![Fig. 12. System composed by \( p \) filters for the generation of trajectories based on uniform B-splines starting from the sequence of the control points \( p_j \).](image)

![Fig. 13. Magnitude of the frequency response of the chain of filters \( M^p(s) \) used for generating uniform B-spline trajectories of degree \( p \).](image)
VI. CONCLUSIONS

Two different design philosophies for online trajectory planners (namely open-loop and closed-loop approaches) have been analyzed. Besides their different structure, the two groups enjoy peculiar features that make each type of generator preferable for a specific application. In particular, open-loop systems have a very simple structure and imply a computational cost lower than similar closed-loop generators. Because of their simple structure, they can be easily generalized in order to implement high order trajectories, with continuous jerk or even higher derivatives. On the other hand, this kind of trajectory planners suffers from some limitations that can be overcome by adopting closed-loop filters. As a matter of fact, filters with a cascade configuration only work with symmetric constraints on velocity, acceleration, etc.. Moreover, the desired bounds cannot be changed in runtime and a new trajectory cannot start before the current motion profile has executed without violating the desired limit values. On the contrary, closed-loop trajectory generators allow to modify the limits in runtime and to start a new motion at any time. However open-loop filters may be preferable in all those applications that are affected by vibrations and resonances, since the linear structure allows a precise frequency characterization of the output trajectory. In the same manner, filters for uniform B-splines generation, that are characterized by a similar structure, can be designed with the purpose of properly shaping their frequency spectrum. Therefore, by selecting the trajectory/filter parameters it is possible to suppress residual vibrations, that may be present because elastic phenomena affecting the robotic system. The effectiveness of the proposed approach is demonstrated by
applying it for the generation of a 3D trajectory to be tracked by a cartesian robot with elastic joints.

REFERENCES


